

CRITERION FOR CONVERGENCE ALMOST EVERYWHERE, with applications

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ABSTRACT.

We derive the necessary and sufficient condition for almost sure convergence of the sequence of measurable functions, and consider some applications in the theory of Fourier series and in the theory of random fields.

Key words and phrases: Measure, sigma-finiteness, sigma field, convergence almost everywhere (almost surely), partition, random variable (r.v.), random processes (r.p.) and random fields (r.f.), separable Banach space, functional, sub-linearity, upper limit, Grand Lebesgue Spaces (GLS), Dirichlet kernel, critical functional, approximation, kernel of functional, criterion, probability, distribution, Orlicz space and Orlicz norm convergence.

1 Introduction, Notations. Statement of problem.

Let $(X = \{x\}, M, \mu)$ be measurable space equipped with sigma-finite non zero measure μ , $(B = \{b\}, \|\cdot\| = \|\cdot\|_B)$ be a (complete) separable Banach space relative the norm function $\|\cdot\| = \|\cdot\|_B$, not necessary to be separable (or reflexive).

Let also

$$F = \{ f(x) = f_\infty(x), \{f_n(x)\}, n = 1, 2, \dots; x \in X \}$$

be a numbered family of measurable functions from the set X into the space B :

$$f_n : X \rightarrow B, n \in \{\infty\} \cup \{1, 2, \dots\}.$$

Our goal in this short article is finding of the necessary and sufficient condition on the family F in the integrals terms for almost sure convergence of the sequence $f_n \rightarrow f_\infty$ or simple $\exists \lim_{n \rightarrow \infty} f_n$ a.e. :

$$\mu\{x, x \in X, \lim_{n \rightarrow \infty} f_n(x) \neq f(x)\} = 0. \quad (1.0)$$

Remark 1.1. We imply in the equality (1.0) that if the limit $\lim_{n \rightarrow \infty} f_n(x)$ can not exists, but on the set with zero measure.

Immediate predecessor of general case besides the special cases: martingales, monotone sequences etc. is the preprint [36]. We intend to generalize the results obtained therein.

There are many applications of solution of this problem in the theory of Fourier series (and integrals) [31], [40] and other orthogonal ones [1], theory of Probability, [3], [4], in particular, in the theory of martingales [19], statistics [38] etc.

2 Main Result: Convergence of a Sequence of a measurable Functions almost everywhere.

Lemma 2.1. There exists a *probabilistic* measure ν defined on at the same sigma-field M which is equivalent to the source measure μ in the Radon-Nikodym sense, i.e. such that the following implication there holds

$$\forall A \in M \Rightarrow [\mu(A) = 0 \Leftrightarrow \nu(A) = 0]. \quad (2.1)$$

Proof. The case of boundedness of the measure $\mu(\cdot)$ is trivial; suppose therefore $\mu(X) = \infty$.

As long as the measure μ is sigma - finite, there exist a countable family of disjoint measurable sets $\{X_m\}$, $m = 1, 2, 3, \dots$, $X_m \in M$, $l \neq m \Rightarrow X_m \cap X_l = \emptyset$, such that

$$0 < \mu(X_m) < \infty, \quad \cup_{m=1}^{\infty} X_m = X, \quad (2.2)$$

so that the family $\{X_m\}$ forms the partition of whole set X .

Define for any set $A \in M$

$$\nu(A) \stackrel{def}{=} \sum_{m=1}^{\infty} \frac{\mu(A \cap X_m)}{2^m \mu(X_m)}. \quad (2.3)$$

Obviously, $\nu(\cdot)$ is sigma-additive probability ($\nu(X) = 1$) measure which is completely equivalent to the initial measure μ .

As a consequence: the μ - complete convergence of the sequence of measurable functions $\{f_n(x)\}$ is equal to the convergence with ν - measure one.

Therefore, we can and will assume without loss of generality that the initial measure μ is probabilistic, i.e. $\mu(X) = 1$.

Further, we can reduce our problem again without loss of generality passing to the sequence $g_n = f_n - f_{\infty}$ to the case when $f_{\infty} = 0$.

A probability language: given the sequence of random variables ξ_n, ξ , $n = 1, 2, \dots$ with values in certain separable Banach space B ; find the necessary and sufficient condition for the following convergence with probability one:

$$\mathbf{P}(\lim_{n \rightarrow \infty} \|\xi_n - \xi\|_B = 0) = 1. \quad (2.4)$$

One can assume as before $\xi = 0$.

Let us introduce the Banach space $c_0(B)$, also separable, consisting on all the sequences $y = \{y(n)\}$ with values in the space B and converging to zero in this space:

$$\|y_n\|_B \rightarrow 0, \quad n \rightarrow \infty,$$

with the norm

$$\|y\|_{c_0(B)} \stackrel{\text{def}}{=} \sup_n \|y_n\|_B.$$

The equality (2.4) may be reformulated as follows: under what necessary and sufficient conditions (criterion)

$$\mathbf{P}(\{\xi_n\} \in c_0(B)) = 1. \quad (2.5)$$

On the other words, we raise the question of finding of necessary and sufficient conditions for convergence of random elements in (separable) Banach spaces, in the terms of a famous monograph of V.V.Buldygin [5].

Let us return to the initial notations: $\{f_n(x)\}$, $\nu(\cdot)$ and so one. Suppose $f(x) = f_\infty(x) = 0$.

We need to introduce some new notations.

$$\tilde{f}_n(x) := \arctan f_n(x), \quad \kappa_n^m = \kappa_n^m(F) \stackrel{\text{def}}{=} \int_X \arctan(\max_{k=n}^m |f_k(x)|) \nu(dx) =$$

$$\int_X \max_{k=n}^m |\tilde{f}_k(x)| \nu(dx), \quad m \geq n + 1; \quad (2.6)$$

$$\overline{\kappa}(F) \stackrel{\text{def}}{=} \overline{\lim}_{n \rightarrow \infty} \sup_{m \geq n+1} \kappa_n^m(F). \quad (2.7)$$

Definition 2.1. The introduced above functional $\overline{\kappa}(F)$ and the like functionals that will be appear further, are sub-linear and will be named as "critical functional".

Theorem 2.1. *In order to $\nu\{x : \lim_{n \rightarrow \infty} f_n(x) = 0\} = 1$, is necessary and sufficient that the family F belongs to the kernel of the critical functional $\overline{\kappa}(F)$:*

$$\lim_{n \rightarrow \infty} \sup_{m > n} \kappa_n^m(F) = 0 \quad (2.8)$$

or equally

$$\overline{\kappa}(F) = 0 \quad \text{or} \quad \text{equally} \quad F \in \ker(\overline{\kappa}). \quad (2.8a)$$

Proof. Necessity.

Let $\nu\{x : (\lim_{n \rightarrow \infty} |f_n(x)| = 0)\} = 1$, then with at the same value of the ν - measure

$$\lim_{n \rightarrow \infty} \sup_{m > n} |f_m(x)| = 0$$

and a fortiori

$$\lim_{n \rightarrow \infty} \sup_{m > n} \arctan |f_m(x)| = 0$$

$\nu(\cdot)$ almost surely. We conclude on the basis of the dominated convergence theorem

$$\lim_{n \rightarrow \infty} \int_X \sup_{m > n} \arctan |f_m(x)| \nu(dx) = 0,$$

which is quite equivalent to the equality (2.8.)

Proof. Sufficiency.

0. We set primarily

$$A = \{x, x \in X : \lim_{n \rightarrow \infty} f_n(x) = 0\} = \{x \in X : \lim_{n \rightarrow \infty} \tilde{f}_n(x) = 0\},$$

$$A_Q = \{x, x \in X : \forall s = 1, 2, \dots \exists N = 1, 2, \dots : \max_{k \in [N, N+Q]} |\tilde{f}_k(x)| < 1/s\}.$$

Obviously,

$$\nu(A) = \lim_{Q \rightarrow \infty} \nu(A_Q).$$

1. Note first of all that

$$\begin{aligned} A &= \{x, x \in X : \lim_{n \rightarrow \infty} |f_n(x)| = 0\} = \{x : \lim_{n \rightarrow \infty} |\tilde{f}_n(x)| = 0\} = \\ &= \bigcap_s \bigcup_N \{x : \sup_{n \geq N} |\tilde{f}_n(x)| < 1/s\} \end{aligned} \quad (2.9)$$

and correspondingly

$$A_Q := \bigcap_s \bigcup_N \{x : \max_{n \in [N, N+Q]} |\tilde{f}_n(x)| < 1/s\}.$$

2. Let the condition (2.8) be satisfied. We consider a supplementary set

$$B := \overline{A} = X \setminus A, \quad B_Q := \overline{A_Q} = X \setminus A_Q. \quad (2.10)$$

Elucidation: the set of elementary events B may contains (theoretically) also the points when the limit does not exists.

We can write

$$B = \cup_s \cap_N \{x : \sup_{n \geq N} |\tilde{f}_n(x)| \geq 1/s\}, \quad (2.11)$$

$$B_Q = \cup_s \cap_N \{x : \max_{n \in [N, N+Q]} |\tilde{f}_n(x)| \geq 1/s\} = \cup_s C_{s,Q}, \quad (2.12)$$

where

$$C_{s,Q} = \cap_N \{x : \max_{n \in [N, N+Q]} |\tilde{f}_n(x)| \geq 1/s\} = \cap_N D_{s,Q}^{(N)}, \quad (2.13)$$

$$D_{s,Q}^{(N)} := \{x, x \in X : \max_{n \in [N, N+Q]} |\tilde{f}_n(x)| \geq 1/s\}. \quad (2.14)$$

3. We obtain using the Tchebychev's inequality:

$$\nu\{ (D_{s,Q}^{(N)}) \} \leq \frac{\kappa_N^{N+Q}}{\arctan(1/s)} \rightarrow 0, \quad N \rightarrow \infty,$$

therefore for all the natural values s, Q

$$\nu(C_{s,Q}) = 0,$$

following

$$\forall Q = 1, 2, \dots \Rightarrow \nu(B_Q) = 0. \quad (2.15)$$

4. We find

$$\nu(B) = \lim_{Q \rightarrow \infty} \nu(B_Q) = 0, \quad (2.16)$$

and ultimately $\nu(A) = 1$, Q.E.D.

Remark 2.1.

We will consider here the case of the Banach space c consisting on all the numerical sequences $\{x(n)\}$ with existing the limit

$$\exists \lim_{n \rightarrow \infty} x(n) =: x(\infty)$$

with at the same norm as above. As before, we consider the classical problem: let $\xi = \{\xi(n)\}$ be a random sequence; find the conditions (necessary conditions and sufficient conditions) under which $\mathbf{P}(\exists \lim_{n \rightarrow \infty} \xi(n)) = 1$ or equally $\mathbf{P}(\{\xi\} \in c) = 1$.

Notations:

$$\tilde{\xi}(n) := \arctan \xi(n), \quad \gamma_n^m = \gamma_n^m(\xi) \stackrel{def}{=} \mathbf{E} \arctan(\max_{k=n}^m |\xi(k) - \xi(n)|).$$

$$\overline{\gamma}(\xi) := \overline{\lim}_{n \rightarrow \infty} \sup_{m \geq n+1} \gamma_n^m(\xi).$$

Proposition 2.1. We find analogously to the theorem 2.1: $\mathbf{P}(\{\xi\} \in c) = 1$ if and only if

$$\lim_{n \rightarrow \infty} \sup_{m \geq n+1} \gamma_n^m(\xi) = 0, \quad (2.17)$$

or briefly

$$\overline{\gamma}(\xi) = 0, \quad \text{or equally} \quad \xi(\cdot) \in \ker \overline{\gamma}. \quad (2.17a)$$

Remark 2.2.

It is known in the theory of martingales, see e.g. [6], [7], [19], chapters 2,3, [39] that the estimation of the maximum distribution play a very important role for the investigation of limit theorems, non asymptotical estimations etc.

It follows from our considerations that at the same is true in more general case of non-martingale processes and sequences.

Remark 2.3. Roughly speaking, the result of theorem 2.1 may be reformulated as follows. Let again $\{\xi(n)\}$ be a sequence of a r.v., $\tilde{\xi}(n) = \arctan(|\xi(n)|)$, and

$$\eta(n) = \sup_{m \geq n} |\tilde{\xi}(m)|.$$

Then

$$\{\omega : \xi(n) \rightarrow 0\} = \{\omega : \eta(n) \rightarrow 0\}.$$

But the random sequence $\{\eta(n)\}$ is monotonically non-increasing, therefore the sequence $\{\eta(n)\}$ tends to zero *with probability one* iff this sequence tends to zero *in probability*, or equally

$$\lim_{n \rightarrow \infty} \mathbf{E}\eta(n) = 0, \quad (2.18)$$

because the variables $\eta(n)$ are uniformly bounded.

Remark 2.4. Let's turn our attention to the properties of introduced above critical functional $\overline{\kappa}(F)$ and $\overline{\gamma}(\xi)$ and so one. Naturally and obviously, the kernels of these functionals are *closed* linear subspaces; if for definiteness $\overline{\gamma}(\xi_1) = 0$ and $\overline{\gamma}(\xi_2) = 0$, then

$$\overline{\gamma}(c_1\xi_1 + c_2\xi_2) = 0, \quad c_1, c_2 = \text{const}. \quad (2.19)$$

3 Almost everywhere convergence of Fourier series.

Let T be a segment $T = [0, 2\pi]$ and μ is customary *renormed* Lebesgue measure $d\mu = dx/(2\pi)$.

Let also $g = g(x)$, $x \in R$ be measurable and integrable (2π) – periodical numerical function. Denote by $s_n(x) = s_n[g](x)$ the its n^{th} partial sum of ordinary Fourier (trigonometrical) series, which may be written through (2π) – periodical convolution with Dirichlet kernel $D_n(x)$:

$$s_n[g](x) = g * D_n(x) = \int_0^{2\pi} D_n(x-y) g(y) dy, \quad (3.1)$$

where

$$D_n(x) = \frac{\sin[(n+1/2)x]}{2\pi \sin(x/2)}. \quad (3.2)$$

The problem of finding (sufficient) conditions under the function $g(\cdot)$ for the almost everywhere convergence

$$s_n[g](x) \xrightarrow[\mu]{\text{a.e.}} g(x) \quad (3.3)$$

is named ordinary as *Luzin's problem* and has a long history, see for example [2], [8], [18], [20], [31], [40], [49].

The following famous result belongs to N.Y.Antonov [2]: if

$$\int_T |f(x)| \cdot \ln^+ |f(x)| \cdot \ln^+ \ln^+ \ln^+ |f(x)| dx < \infty, \quad (3.4)$$

where

$$\ln^+ z = \max(e, \ln z), \quad z > 0,$$

or equally $f(\cdot) \in L \ln L \ln \ln \ln L$, then the convergence (3.3) holds true.

Define also a difference Dirichlet kernel $D_{m,n}(x) = D_m(x) - D_n(x)$, $m \geq n+1$, $n \geq 2$, then

$$D_{m,n}(x) = \pi^{-1} \frac{\sin[(m-n)x/2] \cdot \cos[(m+n+1)x/2]}{\sin(x/2)},$$

and introduce the difference of the Fourier sums

$$s_{m,n}(x) := [D_{m,n} * g](x) = s_m[g](x) - s_n[g](x).$$

Denote also

$$\theta_n^m[g] \stackrel{\text{def}}{=} \int_{[0,2\pi]} \max_{k \in [n+1,m]} \arctan \max_{x \in [0,2\pi]} |s_{k,n}(x)| dx, \quad (3.5)$$

$$\bar{\theta}[g] \stackrel{\text{def}}{=} \overline{\lim}_{n \rightarrow \infty} \sup_{m \geq n+1} \theta_n^m[g]. \quad (3.5a)$$

The functional $g \rightarrow \bar{\theta}[g]$ is now the critical sub-linear functional for considered here problem.

It follows immediately from Theorem 2.1 and Proposition 2.1

Proposition 3.1. The Fourier sums $s_n[g](x)$ for the function $g = g(x)$ converges almost surely to the function $g = g(x)$ (3.3) if and only if

$$\lim_{n \rightarrow \infty} \sup_{m \geq n+1} \theta_n^m[g] = 0. \quad (3.6)$$

or for brevity

$$\bar{\theta}(g) = 0 \quad \text{or} \quad \text{equally} \quad g \in \ker(\bar{\theta}). \quad (3.6a)$$

4 About trial function.

We used in the second section a trial function $x \rightarrow \arctan(|x|)$. Evidently, it can apply some another functions.

Definition 4.1. We will denote by KB the class of all numerical functions $\{\phi\}$, $\phi : R \rightarrow R_+$ satisfying the following conditions:

$$\mathbf{A.} \quad \phi(x) \geq 0; \quad \phi(x) = 0 \Leftrightarrow x = 0,$$

the condition of positivity;

$$\mathbf{B.} \quad 0 < x < y \Rightarrow \phi(x) < \phi(y),$$

the strong monotonicity on the right - hand real axis;

C. Continuity: function $x \rightarrow \phi(x)$ is continuous on the whole axis R .

D. The function $\phi(x)$ is even: $\forall x \in R \Rightarrow \phi(-x) = \phi(x)$.

$$\mathbf{E.} \quad \sup_{x \in R} \phi(x) < \infty,$$

the condition of boundedness.

For example:

$$\phi(x) = \arctan |x|, \quad \phi(x) = \frac{|x|}{1 + |x|}, \quad \phi(x) = \frac{x^2}{1 + x^2}$$

and so one.

The assertion of theorem 2.1 may be rewritten as follows. Denote as before for any function ϕ from the set KB

$$\kappa_n^m(\phi) = \kappa_n^m(\phi, F) \stackrel{\text{def}}{=} \int_X \phi(\max_{k=n}^m |f_k(x)|) \nu(dx), \quad m \geq n + 1, \quad (4.1)$$

$$\overline{\kappa}(\phi) = \overline{\kappa}(\phi, F) \stackrel{\text{def}}{=} \overline{\lim}_{n \rightarrow \infty} \sup_{m \geq n+1} \kappa_n^m(\phi, F). \quad (4.1a)$$

Theorem 4.1. In order to $\nu\{x : \lim_{n \rightarrow \infty} f_n(x) = 0\} = 1$, it is sufficient that for *some* function $\phi(\cdot)$ from the class KB

$$\lim_{n \rightarrow \infty} \sup_{m \geq n+1} \kappa_n^m(\phi, F) = 0, \quad (4.2)$$

or equally

$$\{F\} \in \ker(\overline{\kappa}(\phi)). \quad (4.2a)$$

and is necessary that for *arbitrary* function ϕ belonging to at the same set KB the relation (4.2) or (4.2a) holds true.

Definition 4.2. We will denote by K the class of all numerical functions $\{\phi\}$, $\phi : R \rightarrow R_+$ satisfying the foregoing conditions at this section **A**, **B**, **C**, **D**, i.e. all the conditions except for the latter condition of boundedness **E**. For instance, let $\phi_p(x) := |x|^p$, $p = \text{const} > 0$; then $\phi_p(\cdot) \in K$.

In particular, arbitrary Young-Orlicz function $\phi(x)$ belongs to the set K .

Theorem 4.2. In order to $\nu\{x : \lim_{n \rightarrow \infty} f_n(x) = 0\} = 1$, it is sufficient that for some function $\phi(\cdot)$ from the class K

$$\lim_{n \rightarrow \infty} \sup_{m \geq n+1} \kappa_n^m(\phi, F) = 0, \quad (4.3)$$

and herewith

$$\lim_{n \rightarrow \infty} \int_X \phi(\|f_n(x)\|B) \nu(dx) = 0; \quad (4.4a)$$

$$\sup_n \int_X \phi(\|f_n(x)\|B) \nu(dx) \leq \sup_n \sup_{m \geq n+1} \kappa_n^m(\phi, F). \quad (4.4b)$$

The relation (4.4a) implies on the language of the theory of Orlicz's function the so-called moment, or weak convergence $f_n \rightarrow 0$ in the Orlicz norm $||| \cdot |||_{L\phi}$; the last equality (4.4b) denotes the uniform boundedness of the considered sequence $\{f_n(\cdot)\}$ in this space.

If in addition this function $\phi = \phi(z)$ is Young-Orlicz function satisfying the so-called Δ_2 condition, then the sequence of functions $f_n(\cdot)$ convergent to zero also in the Orlicz's norm $||| \cdot |||_{L\phi}$:

$$\lim_{n \rightarrow \infty} ||| f_n |||_{L\phi} = 0. \quad (4.5)$$

Let us show another approach which is closely to the so-called Grand Lebesgue Spaces (GLS), see e.g. [14], [22], [27], [30], [33], [37].

Let again $F = \{f_n(x)\}$, $x \in X$ be as before in the first section be the sequence of measurable functions. Define a new function

$$\psi(p) \stackrel{\text{def}}{=} \sup_n \left[\int_X |f_n(x)|^p \nu(dx) \right]^{1/p}, \quad (4.6)$$

the so-called *natural* function for the sequence $F = \{f_n(x)\}$, $x \in X$, and suppose its finiteness for certain interval of the form $1 \leq p < R$, where $1 < R = \text{const} \leq \infty$.

This function $\psi = \psi(p)$ generated the so - called Grand Lebesgue Space (GLS) $G\psi$ consisting on all the numerical measurable functions $h = h(x)$, $x \in X$ with finite norm

$$\|h(\cdot)\|_{G\psi} := \sup_{p \in (1, R)} \left[\frac{|h|_p}{\psi(p)} \right], \quad (4.7)$$

where as usually

$$|h|_p := \left[\int_X |h(x)|^p \nu(dx) \right]^{1/p}.$$

These spaces are complete rearrangement invariant Banach spaces which are detail investigated in [27], [30], [33] and so one.

Define the following critical functions

$$\begin{aligned} \lambda_n^m(F) &= \left\| \max_{k=n+1}^m |f_k(\cdot)| \right\|_{G\psi}, \\ \overline{\lambda}(F) &= \overline{\lim}_{n \rightarrow \infty} \sup_{m \geq n+1} \lambda_n^m(F). \end{aligned} \quad (4.8)$$

We conclude as before:

Theorem 4.2. If $\overline{\lambda}(F) = 0$, then the sequence $f_n(x)$ converges to zero almost everywhere and in the Grand Lebesgue Space norm $G\psi$.

As long as the classical Lebesgue-Riesz spaces $L_p(X)$, $p = \text{const} \geq 1$ are the extremal case for GLS spaces, and also the particular cases of the classical Orlicz spaces, we conclude denoting again for $f = \{f_n(x)\}$, $x \in X$

$$\overline{l}_p(F) = \overline{\lim}_{n \rightarrow \infty} \sup_{m \geq n+1} |f_n - f_m|_p. \quad (4.9)$$

Theorem 4.3. If $\overline{l}_p(F) = 0$, then the sequence $f_n(x)$ converges almost everywhere as well as in the Lebesgue-Riesz norm $L_p(X)$.

5 Convergence of random elements in separable Banach spaces.

We return to the raised before the question of finding of necessary and sufficient conditions for convergence of random elements in separable Banach spaces, see [5].

The particular case of this problem is the problem of continuity of random processes and fields, is considered in the articles and books [10], [11], [12], [13], [16], [17], [21], [24], [26], [27], [28], [29], [32], [33], [36], [41], [42], [43], [44], [45], [46], [48] etc.

This problem may be easily reduced to the problem of uniform convergence of random numerical functions, for instance, the problem of uniform convergence of expression of the series by Franklin orthogonal system, see [9], [15], [36].

We consider here the problem of convergence of random elements with values in Banach space.

In detail: let $\zeta = \{\zeta(n)\}$ be a random sequence with values in the separable Banach space B ; find the conditions (necessary conditions and sufficient conditions) under which $\mathbf{P}(\exists \lim_{n \rightarrow \infty} \zeta(n)) = 1$ or equally $\mathbf{P}(\{\zeta\} \in c(B)) = 1$.

Notations:

$$\tilde{\zeta}(n) := \arctan \zeta(n), \quad \tau_n^m = \tau_n^m(\xi) \stackrel{\text{def}}{=} \mathbf{E} \arctan(\max_{k=n}^m \|\zeta(k) - \zeta(n)\|_B), \quad (5.1)$$

$$\bar{\tau}(\zeta) := \overline{\lim}_{n \rightarrow \infty} \sup_{m \geq n+1} \tau_n^m(\zeta). \quad (5.1a)$$

Proposition 5.1. We conclude analogously to the theorem 2.1: $\mathbf{P}(\{\zeta\} \in c) = 1$ if and only if

$$\lim_{n \rightarrow \infty} \sup_{m \geq n+1} \tau_n^m(\xi) = 0, \quad (5.2)$$

or briefly

$$\bar{\tau}(\zeta) = 0, \quad \text{or equally} \quad \zeta(\cdot) \in \ker \bar{\tau}. \quad (5.3)$$

6 Concluding remarks.

I. The case of metric (linear) space B .

It is not hard to generalize obtained above results on the case when the Banach space B is replaced by certain separable linear metric space L equipped with translation invariant metric function $\rho = \rho(x - y)$.

II. Recall that for the convergence in probability (measure) of the sequence of Banach space valued r.v. η_n the necessary and sufficient condition is following

$$\lim_{n, m \rightarrow \infty} \mathbf{E} \arctan \|\eta_n - \eta_m\|_B = 0.$$

III. It is interest by our opinion to investigate in the spirit of this article the case of the non-sequential convergence; as well as to obtain the criterion for a.e. convergence for multiple sequences, especially multiple Fourier series.

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